A Beam Dynamics View on a Generalized Formulation of Spin Dynamics, Based on Topological Algebra, with Examples.

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FLASH
Free–Electron Laser in Hamburg

Abstract

Here I rephrase the results of work[1, 2, 3, 4, 5, 6] performed in several collaborations with **K.Heinemann**[1], **J.A.Ellison**[1], **D.P.Barber**[2], **and A.Kling**[3] on a generalized look on spin dynamics and beam polarization in storage rings. It is done in a way that emphasizes the applicability of the concepts to real world polarized beams rather than presenting the results in their most general form. The latter view can be found in several articles on the ArXiv and will be published in refereed journals soon. I will introduce several "spin-related" systems, state some selected main results of the above mentioned work and then recover and compare some basic (and some not so basic) findings for the various systems in the light of our generalized approach.

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References

[1] K.Heinemann: "Two topics in particle accelerator beams", *Doctoral Thesis*, University of New Mexico, Albuquerque, NM, USA (2010) `http://repository.unm.edu/bitstream/handle/1928/10879/`→ → `border1.pdf ?sequence=3&isAllowed=y`

[2] K.Heinemann, D.P.Barber, J.A.Ellison and M.Vogt: "A detailed and unified treatment of spin-orbit systems using tools distilled from the theory of bundles", DESY Preprint 15-010, also arXiv:1501.02747 [physics.acc-ph] (2015).

[3] K.Heinemann, J.A.Ellison, D.P.Barber, M.Vogt: "A New and Unifying Approach to Spin Dynamics and Beam Polarization in Storage Rings", DESY Preprint 14-163 also arXiv:1409.4373v2 (2014)

[4] K.Heinemann, J.A.Ellison, D.P.Barber, M.Vogt: extended and rewritten version of [3], *submitted* (2017)

[5] D.P.Barber: "The invariant polarisation–tensor field for deuterons in-strage rings and the Bloch equation for the polarisation-tensor density", arXiv:1510.04936 [physics.acc-ph] (2015)

[6] D.P.Barber, A.Kling, M.Vogt: "The uniqueness of the invariant polarisation–tensor field for spin–1 particles in storage rings", arXiv:1510.07963 [physics.acc-ph] (2015)

[7] D.P.Barber, M.Vogt: "Spin Motion at and Near Orbital Resonance in Storage Rings with Siberian Snakes I: At Orbital Resonance", *New Journal of Physics* **8**, 296, November 2006.

[8] J.Pinkman, W.White, T.Salamanca, *in Breaking Bad*, S2-E2 "Grilled", **AMC** (2009 :-)

Introduction

Spin/Orbit Dynamics

- Time–Discrete picture → maps!
- Integrable orbital motion on torus $\mathcal{T}^d$:
  actions $J = \text{const}$ & phases mapped by tune ω
  $$M_\omega : \mathcal{T}^d \to \mathcal{T}^d, \phi \mapsto M_\omega(\phi) = [\phi + \omega]_{\mathcal{T}} \quad (1)$$
- "Polarization" assume BMT–evolution for vector Pol. $\vec{s}$ and tensor Pol. $\underline{\bar{s}}$ starting at ϕ:
  $$\vec{s} \mapsto \underline{R}(\phi)\,\vec{s}\ ,\ \underline{\bar{s}} \mapsto \underline{R}(\phi)\,\underline{\bar{s}}\,\underline{R}(\phi)^{\mathrm{T}}$$
  $$\underline{R} : \mathcal{T}^d \to \mathbf{SO(3)}\ ,\ \Phi \mapsto \underline{R}(\phi) \in \mathbf{SO(3)} \quad (2)$$
  w/ $\vec{s} \in \mathbb{R}^3$, $\underline{\bar{s}} \in \mathbb{R}^{3\times3}$, $\underline{\bar{s}} = \underline{\bar{s}}^{\mathrm{T}}$, $\mathrm{trace}\,\underline{\bar{s}} = 0$
- "Spin" ≡ "Polarization" w/ norm 1:
  $\|\hat{s}\|_2 = 1$ (Eucl. norm) $\Leftarrow \|\underline{R}\,\vec{s}\|_2 = \|\vec{s}\|_2$,
  $\|\underline{\check{s}}\|_{\mathrm{t}} := \sqrt{\mathrm{trace}\,(\underline{\check{s}}^{\mathrm{T}}\underline{\check{s}})} = 1 \Leftarrow \|\underline{R}\,\underline{\bar{s}}\,\underline{R}^{\mathrm{T}}\|_{\mathrm{t}} = \|\underline{\bar{s}}\|_{\mathrm{t}}$

Dynamics of Fields

- Vector/tensor polarization/Spin fields ≡ sequences of fields on the torus: F_n $(:= \vec{S}_n/\hat{S}_n)$ / $\underline{F}_n$ $(:= \underline{\bar{S}}_n/\underline{\check{S}}_n)$, $n \in \mathbb{N}_0$, so that
  $$F_{n+1}(M_\omega(\phi)) = \underline{R}(\phi)\,F_n(\phi) \quad \Leftrightarrow$$
  $$F_{n+1}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\,F_n(M_\omega^{-1}(\phi)) \quad (3)$$
  $$\underline{F}_{n+1}(M_\omega(\phi)) = \underline{R}(\phi)\,\underline{F}_n(\phi)\,\underline{R}(\phi)^{\mathrm{T}} \quad \Leftrightarrow$$
  $$\underline{F}_{n+1}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\,\underline{F}_n(M_\omega^{-1}(\phi))\,\underline{R}(M_\omega^{-1}(\phi))^{\mathrm{T}} \quad (4)$$
  note: e.g. $F_{n+1} = \mathcal{M}_\omega\,\underline{R}\cdot\mathcal{M}_\omega\,F_n$, with $\mathcal{M}_\omega$ the P.F. op. of M_ω!
- The vector/tensor polarization/spin fields are **invariant**, if $F_{n+1} = F_n$, or $\underline{F}_{n+1} = \underline{F}_n$ (→ skip n!)
  $$\text{IvPF: } \vec{P}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\,\vec{P}(M_\omega^{-1}(\phi)) \quad (5)$$
  $$\text{IvSF: } \hat{N}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\,\hat{N}(M_\omega^{-1}(\phi)) \quad (6)$$
  $$\text{ItPF: } \underline{\bar{P}}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\,\underline{\bar{P}}(M_\omega^{-1}(\phi))\,\underline{R}(M_\omega^{-1}(\phi))^{\mathrm{T}} \quad (7)$$
  $$\text{ItSF: } \underline{\check{N}}(\phi) = \underline{R}(M_\omega^{-1}(\phi))\,\underline{\check{N}}(M_\omega^{-1}(\phi))\,\underline{R}(M_\omega^{-1}(\phi))^{\mathrm{T}} \quad (8)$$

← **(Q1) Are these invariants somehow related for common M_ω, $\underline{R}$???**
(Q2) How do they relate for varying M_ω, $\underline{R}$???

- Note: the trivial polarization fields $\vec{P}_{\mathrm{null}}(\phi) \equiv \vec{0}$ & $\underline{\bar{P}}_{\mathrm{null}}(\phi) \equiv \underline{0}$ are always invariant

The New Formalism (Basics)

(a) **$\mathbf{SO(3)}$–Action :**
Let E be a "set" and
$$\boxed{l : \mathbf{SO(3)} \times E \to E,\ (\underline{A}, x) \mapsto y = l(\underline{A}; x) \in E}$$
so that
$$l(\underline{1}; x) = x \quad {\scriptstyle \forall x \in E}$$
$$l(\underline{A}_2\underline{A}_1; x) = l(\underline{A}_2; l(\underline{A}_1; , x)) \quad {\scriptstyle \forall x\in E, \forall \underline{A}_1,\underline{A}_2 \in \mathbf{SO(3)}} \quad (9)$$
then l is the $\mathbf{SO(3)}$–**Action** of the $\mathbf{SO(3)}$–**Space** (E, l).
If E is a lin. space and l is lin. in $\underline{A}$ & x, (E, l) is a **representation**.

→ Vector pola.: $E_{\vec{\mathrm{v}}} := \mathbb{R}^3$, $l_{\vec{\mathrm{v}}}(\underline{A}; \vec{s}) := \underline{A}\vec{s}$
Vector spin : $E_{\hat{\mathrm{v}}} := \mathcal{S}_2$, $l_{\hat{\mathrm{v}}}(\underline{A}; \hat{s}) := \underline{A}\hat{s}$
Tensor pola.:
$E_{\bar{\mathrm{t}}} := \{\underline{\bar{s}} \in \mathbb{R}^{3\times3} : \underline{\bar{s}} = \underline{\bar{s}}^{\mathrm{T}}, \mathrm{trace}\underline{\bar{s}} = 0\}$, $l_{\bar{\mathrm{t}}}(\underline{A}; \underline{\bar{s}}) := \underline{A}\,\underline{\bar{s}}\,\underline{A}^{\mathrm{T}}$
Tensor spin : $E_{\check{\mathrm{t}}} := \{\underline{\check{s}} \in E_{\bar{\mathrm{t}}} : \|\underline{\check{s}}\|_{\mathrm{t}} = 1\}$,
$l_{\check{\mathrm{v}}}(\underline{A}; \underline{\check{s}}) := \underline{A}\,\underline{\check{s}}\,\underline{A}^{\mathrm{T}}$
combined E/orbit–map K of (E, l) with M_ω & $\underline{R}$:
$$\boxed{\mathcal{T}^d \times E \xrightarrow{K} \mathcal{T}^d \times E,\ (\phi, x) \overset{K}{\mapsto} (M_\omega(\phi), l(\underline{R}(\phi); x))}$$
→ **invariant fields:** condition becomes
$$\boxed{F \circ M_\omega = l_{\mathrm{V}}(\underline{R}; F)\ \&\ \underline{F} \circ M_\omega = l_{\mathrm{t}}(\underline{R}; \underline{F})}$$

(b) **(E, l)–Orbit E_x of x :**
$\forall x \in E$: is a subset $E_x \subset E$
$$E_x := l(\mathbf{SO(3)}; x) := \{l(\underline{A}; x) : \underline{A} \in \mathbf{SO(3)}\}$$
→ $\boxed{\text{inv. sets of comb. map: } K(\mathcal{T}^d \times E_x) = \mathcal{T}^d \times E_x}$
→ $l_{\vec{\mathrm{v}}}(\mathbf{SO(3)}; \vec{s}) \equiv \mathcal{S}_2 \cdot \|\vec{s}\|_2$, $l_{\vec{\mathrm{v}}}(\mathbf{SO(3)}; \vec{0}) = \vec{0}$, $l_{\hat{\mathrm{v}}}(\mathbf{SO(3)}; \hat{s}) \equiv \mathcal{S}_2$

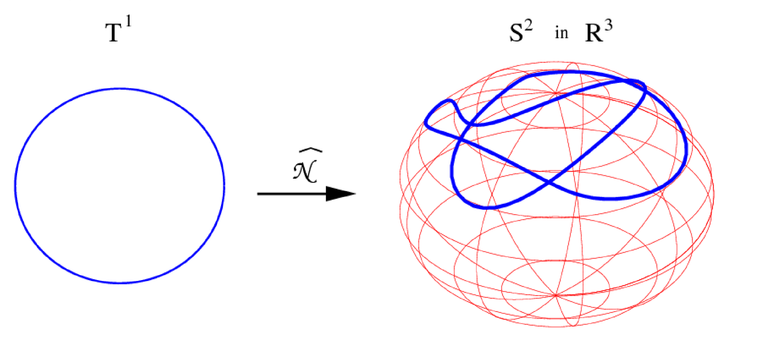

Fig.1: `SPRINT` (HERA–p) example of $\mathcal{C}^0$–IvSF driven by 1-d (vertical) orbital motion.

(c) **Isotropy Group :**
Let (E, l) be $\mathbf{SO(3)}$–space & $x \in E$, the *subgroup* of $\mathbf{SO(3)}$ for which x is a *fixed point* of $l(\underline{A}; \cdot)$ is called isotropy group of (E, l) at x:
$$\boxed{\mathbf{Iso}(E, l; x) := \{\underline{A} \in \mathbf{SO(3)} : l(\underline{A}; x) = x\}}$$
→ $\mathbf{Iso}(E, l; x) = \mathbf{SO(3)}$ iff $E_x = \{x\}$
→ $\mathbf{Iso}(E_{\vec{\mathrm{v}}}, l_{\vec{\mathrm{v}}}; \vec{0}) = \mathbf{SO(3)}$,
$\mathbf{Iso}(E_{\vec{\mathrm{v}}}, l_{\vec{\mathrm{v}}}; \vec{s} \neq \vec{0}) = \{\text{rotations around } \vec{s}\} \cong \mathbf{SO(2)}$

(d) **G–Map (of $\mathbf{SO(3)}$) :**
maps between two $\mathbf{SO(3)}$–spaces
& is structure preserving:
$\Gamma : (E_1, l_1) \to (E_2, l_2)$
$\boxed{l_2(\underline{A}; \Gamma(x)) = \Gamma(l_1(\underline{A}; x))}$,
$\forall \underline{A} \in \mathbf{SO(3)}$, $\forall x \in E_1$

← Don't call it "$\mathbf{SO(3)}$–map", since that smells like
$\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{s} \mapsto \vec{A}(\vec{s}) := \underline{A}\vec{s}$, $\underline{A} \in \mathbf{SO(3)}$!

→ $\boxed{\Gamma_{\check{\mathrm{t}}\leftarrow\hat{\mathrm{v}}} : E_{\hat{\mathrm{v}}} \to E_{\check{\mathrm{t}}},\ \hat{s} \mapsto \sqrt{3/2}(\underline{1} - \tfrac{1}{3}\hat{s}\hat{s}^{\mathrm{T}})}$
fulfills $l_{\check{\mathrm{t}}}(\underline{A}; \Gamma_{\check{\mathrm{t}}\leftarrow\hat{\mathrm{v}}}(\hat{s})) = \Gamma_{\check{\mathrm{t}}\leftarrow\hat{\mathrm{v}}}(l_{\hat{\mathrm{v}}}(\underline{A}; \hat{s}))$

- More examples:
  → **Singlet repres.:** $E_{\mathrm{id}} = \mathbb{R}$, $l_{\mathrm{id}}(\underline{A}; \rho) := \rho$
  → **Liouville PSD:** $\Psi_{n+1} = l_{\mathrm{id}}(\underline{R}; \Psi_n \circ M_\omega^{-1}) \equiv \Psi_n \circ M_\omega^{-1}$
  → **inv. Liouville PSD** $\Psi \circ M_\omega = l_{\mathrm{id}}(\underline{R}; \Psi) \equiv \Psi$
  → **Product action:** $E_{1\times2} = E_1 \times E_2$,
  $l_{1\times2}(\underline{A}; (x_1, x_2)) := (l_1(\underline{A}; x_1), l_2(\underline{A}; x_2))$
- Most of this is from **ABQ**, so we're not surprised...

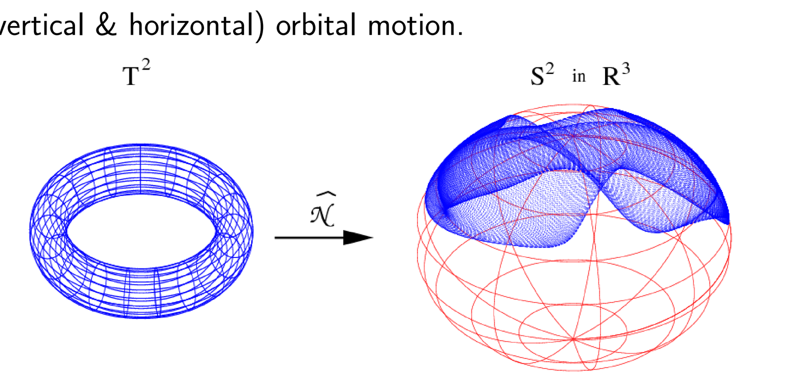

Fig.2: `SPRINT` (HERA–p) example of $\mathcal{C}^0$–IvSF driven by 2-d (vertical & horizontal) orbital motion.

...there's a secret ingredient in it[8]...

- ...and it's neither red nor green chile:
- For the theorems to work we need certain **regularity** constraints:
- We choose **global continuity** (on topological spaces E),
- e.g.: $M_\omega \in \mathrm{Homeo}(\mathcal{T}^d)$, $\underline{R} \in \mathcal{C}^0(\mathcal{T}^d, \mathbf{SO(3)})$ and
- **all** our (invariant) fields (and candidates) need to be **globally continuous**, i.e. $\in \mathcal{C}^0(\mathcal{T}^d, E)$
- I call 'em: $\mathcal{C}^0$**–IvPF**, $\mathcal{C}^0$**–IvSF**, $\mathcal{C}^0$**–ItPF** & $\mathcal{C}^0$**–ItSF** !

Normal Form Theorem (NFT)

Let $\underline{T} \in \mathcal{C}^0(\mathcal{T}^d, \mathbf{SO(3)})$, (E, l), M_ω, $\underline{R}$ as before, and $x \in E$ fixed.
Define $f \in \mathcal{C}^0(\mathcal{T}^d, E)$, $\underline{R}' \in \mathcal{C}^0(\mathcal{T}^d, \mathbf{SO(3)})$ by
$$f := l(\underline{T}; x)\ ,\ \underline{R}' := \underline{T}^{\mathrm{T}} \circ M_\omega\,\underline{R}\,\underline{T}\ . \quad (10)$$
Then f is an invariant (E, l)–field $(f \circ M_\omega = l(\underline{R}; f))$, iff
$$\underline{R}'(\phi) \in \mathbf{Iso}(E, l; x)\ \forall \phi \in \mathcal{T}^d\ . \quad (11)$$

- If $\underline{R}' \in$ some subgroup of $\mathbf{SO(3)}$
  ⇒ $(M_\omega, \underline{R}')$ is a normal form of $(M_\omega, \underline{R})$
- $(E_{\hat{\mathrm{v}}}, l_{\hat{\mathrm{v}}})$ w/ $\hat{z} := (0, 0, 1)^{\mathrm{T}}$:
  $\boxed{\hat{N} := l_{\hat{\mathrm{v}}}(\underline{T}; \hat{z}) \text{ is a } \mathcal{C}^0\text{–IvSF iff } \underline{R}'(\phi) \in \mathbf{SO(2)}}$

Decomposition Theorems

- **$\mathbf{SO(3)}$ Mapping Lemma (SML):**
  Let $\Gamma \in \mathcal{C}^0((E_1, l_1), (E_2, l_2))$ be a **G**–map, $f_1 \in \mathcal{C}^0(\mathcal{T}^d, E_1)$, and $f_2 \in \mathcal{C}^0(\mathcal{T}^d, E_2)$ be defined by $f_2 := \Gamma \circ f_1$. Then
  $l_2(\underline{R} \circ M_\omega^{-1}; f_2 \circ M_\omega^{-1}) = \Gamma(l_1(\underline{R} \circ M_\omega^{-1}; f_1 \circ M_\omega^{-1}))$,
  for all M_ω, $\underline{R}$, i.e. the field dynamics is preserved.
- i.p.: $\boxed{f_1 \circ M_\omega = l_1(\underline{R}, f_1) \Rightarrow f_2 \circ M_\omega = l_2(\underline{R}, f_2)}$
- if $\Gamma \in \mathrm{Homeo}((E_1, l_1), (E_2, l_2))$,
  then also "⇐" is true.
- **Decomposition Corollary (DC):**
  The DC generalizes the SML to **G**–maps from (E_1, l_1)–orbit E_{1,x_1} to (E_2, l_2)–orbit E_{2,x_2} of $x_1 \in E_1$, $x_2 \in E_2$.

Remarks

- The NFT answers, to some extent **(Q2)**,
  while the SML/DC answer **(Q1)**.
- The proofs can be found in [1, 2, 4].
- The above sources state further theorems.
- This poster resembles a reduction to what I think are the *highlights* of our results.
- **Global** Continuity is a *strong* restriction (see 4th example).
- Global **Continuity** is a *weak* restriction, since functions realized in physics normally tend to be (piecewise) **smooth** ($\mathcal{C}^\infty$)
- The regularity constraints for our framework could be made stronger (→ globally $\mathcal{C}^k$, $k > 0$) or weaker (→ globally measurable), thereby modifying the applicability of the premises and the strength of the conclusions.

⇒ Some findings might get strengthened when stronger constraining regularity, some might get weakened when weakening the constraints, some might turn out robust.

Example 1: Relation $\mathcal{C}^0$–IvSF ↔ $\mathcal{C}^0$–ItSF

- $\Gamma_{\check{\mathrm{t}}\leftarrow\hat{\mathrm{v}}}$ (see above) is a **G**–map in $\mathcal{C}^0(E_{\hat{\mathrm{v}}}, E_{\check{\mathrm{t}}})$

⇒ $\underline{\check{N}} := \Gamma_{\check{\mathrm{t}}\leftarrow\hat{\mathrm{v}}}(\hat{N})$ is a $\mathcal{C}^0$–ItSF, if $\hat{N}$ is a $\mathcal{C}^0$–IvSF.
←The constructed $\mathcal{C}^0$–ItSF has 2 distinct eigenval's.

⇒ If the $\mathcal{C}^0$–IvSF is unique up to global sign, so is the $\mathcal{C}^0$–ItSF [6].

- To construct $\underline{\check{S}}$'s that have 3 distinct eigenval's:
  $\Gamma^{(\alpha,\beta)}_{3\mathrm{ev}} : E_{\hat{\mathrm{v}}} \times E_{\check{\mathrm{t}}} \to E_{\check{\mathrm{t}}}$,
  $(\hat{f}, \hat{g}) \mapsto \alpha\underline{1} - (2\alpha + \beta)\hat{f}\hat{f}^{\mathrm{T}} + (\beta - \alpha)\hat{g}\hat{g}^{\mathrm{T}}$ w/
  $\alpha^2 + \alpha\beta + \beta^2 = 1/2$ is a **G**–map in $\mathcal{C}^0(E_{\hat{\mathrm{v}}} \times E_{\hat{\mathrm{v}}}, E_{\check{\mathrm{t}}})$.
  We have shown in[6] that $\Gamma^{(\alpha,\beta)}_{3\mathrm{ev}}$ can only generate a $\mathcal{C}^0$–ItSF, when the system is on **spin–orbit resonance**, i.e. when the $\mathcal{C}^0$–IvSF is non–unique!
- If a $\mathcal{C}^0$–ItPF has only 1 eigenvalue, it must be the trivial one.

Example 2: Spin–1/2 Density Matrix

- The physics–interface between the macroscopic, classical description of a particle beam in an accelerator, and a QM/QFT scattering processes is the **density matrix** $\underline{\rho}$. ($\underline{\rho}^{1/2}$ f. spin–1/2, $\underline{\rho}^{1}$ f. spin–1)
- Here: $\underline{\rho}_J^{1/2}(\phi)$ for given torus w/ fixed orbital actions $J = \text{const}$
  $$\underline{\rho}_J^{1/2}(\phi) := \Psi_J(\phi)\tfrac{1}{2}\left(\underline{1} + \underline{\vec{\sigma}} \cdot \vec{S}_J(\phi)\right)$$
  w/ Ψ_J the (orbital) Liouville PSD, $\vec{S}_J$ the pola. field, both describing the beam, and $\underline{\vec{\sigma}}$ is the vector of Pauli matrices.
- $\underline{\rho}^{1/2} \in E_{1/2} := \{\underline{r} \in \mathbb{C}^{2\times2} : \underline{r}^\dagger = \underline{r}\}$
- $\Gamma_{1/2} : E_{\mathrm{id}} \times E_{\vec{\mathrm{v}}} \to E_{1/2}$, $(\psi, \vec{s}) \mapsto \tfrac{1}{2}(\psi\underline{1} + \underline{\vec{\sigma}} \cdot \vec{s})$
  $\underline{\rho}^{1/2} = \Gamma_{1/2}(\Psi, \Psi\vec{S})$
- $\Gamma_{1/2} \in \mathrm{Homeo}(E_{\mathrm{id}} \times E_{\vec{\mathrm{v}}}, E_{1/2})$ is **G**–map

⇒ $\underline{\rho}^{1/2}_{\mathrm{equi}} = \Gamma_{1/2}(\Psi_{\mathrm{equi}}, \Psi_{\mathrm{equi}}\vec{P})$ is **inv. $(E_{1/2}, l_{1/2})$–field iff** Ψ_{equi} is an inv. Liouv. PSD **and** $\vec{P}$ is a $\mathcal{C}^0$–IvPF.

Example 3: Spin–1 Density Matrix

- $\underline{\rho}^1 := \Psi\tfrac{1}{3}\left(\underline{1} + \tfrac{3}{2}\underline{\vec{\Sigma}} \cdot \vec{S} + \sqrt{\tfrac{3}{2}}\sum_{i,j=1}^{3} \underline{\bar{S}}_{ij}(\underline{\Sigma}_i\,\underline{\Sigma}_j + \underline{\Sigma}_j\,\underline{\Sigma}_i)\right)$
  $$\underline{\Sigma}_{1,2,3} := \sqrt{\tfrac{1}{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
- $\underline{\rho}^1 \in E_1 := \{\underline{r} \in \mathbb{C}^{3\times3} : \underline{r}^\dagger = \underline{r}\}$
- $\Gamma_1 : E_{\mathrm{id}} \times E_{\vec{\mathrm{v}}} \times E_{\bar{\mathrm{t}}} \to E_1$, $(\psi, \vec{s}, \underline{\bar{s}}) \mapsto$
  $\mapsto \tfrac{1}{3}\left(\psi\underline{1} + \tfrac{3}{2}\underline{\vec{\Sigma}} \cdot \vec{s} + \sqrt{\tfrac{3}{2}}\sum_{i,j=1}^{3} \underline{\bar{s}}_{ij}(\underline{\Sigma}_i\,\underline{\Sigma}_j + \underline{\Sigma}_j\,\underline{\Sigma}_i)\right)$
  $\underline{\rho}^1 = \Gamma_1(\Psi, \Psi\vec{S}, \Psi\underline{\bar{S}})$
- $\Gamma_1 \in \mathrm{Homeo}(E_{\mathrm{id}} \times E_{\vec{\mathrm{v}}} \times E_{\bar{\mathrm{t}}}, E_1)$ is **G**–map

⇒ $\underline{\rho}^1_{\mathrm{equi}} = \Gamma_1(\Psi_{\mathrm{equi}}, \Psi_{\mathrm{equi}}\vec{P}, \Psi_{\mathrm{equi}}\underline{\bar{P}})$ is **inv. (E_1, l_1)–field iff** Ψ_{equi} is an inv. Liouv. PSD **and** $\vec{P}$ is a $\mathcal{C}^0$–IvPF **and** $\underline{\bar{P}}$ is a $\mathcal{C}^0$–ItPF.

- The maximum attainable equilibrium polarization state is realized (for spin–1/2 & 1), when
  $\underline{\rho}^{1/2}_{\mathrm{equi}} \to \Gamma_{1/2}(\Psi_{\mathrm{equi}}, \Psi_{\mathrm{equi}}\hat{N})$ (i.e. $\hat{N}$ is $\mathcal{C}^0$–IvSF) &
  $\underline{\rho}^1_{\mathrm{equi}} \to \Gamma_1(\Psi_{\mathrm{equi}}, \Psi_{\mathrm{equi}}\hat{N}, \Psi_{\mathrm{equi}}\underline{\check{N}})$ (and $\underline{\check{N}}$ is $\mathcal{C}^0$–ItSF)

A Discontinuous Example (4)

- Slightly artificial set up:
- M_ω resonant: $\frac{\omega_y}{2\pi} = \frac{1}{4n-2}$, $n = 1, 2, 3, \ldots$
- $\underline{R}$ given by **Single Resonance Model** & added **Lee–Courant** 2 snake scheme =
  → 2 Siberian snakes 180° in azimuth apart,
  → both snake axes in the ring–plane,
  → axes perpendicular
- IvSF needs $2n$ discontinuities (sign flips) in ϕ_y [2, 7]
  (otherwise it becomes twin–valued under iteration of $\underline{R}$!)
- IvSF is not $\mathcal{C}^0$–IvSF
  ⇒ **framework does not apply**
- However, corresponding ItSF **is** $\mathcal{C}^0$–ItSF
  ⇒ **framework does apply**
  ← except *example 1*.

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